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FURTHER PROPERTIES OF THE GENERAL INTEGRAL.

By P. J. Daniell.

Introduction. In these Annals there appeared a paper by the author on a general form of integral of which the Lebesgue integral and the Radon-Young integral are special cases.* That paper confined itself more particularly to the definitions and proofs of existence. In this paper some special points are discussed which bring the theory into closer relation with the Lebesgue theory. It is advisable first to give a resumé of the foundations of our general theory.

We consider functions $f(p)$ of general elements, p. A class T_0 of such functions is assumed which is closed with respect to the operations, multiplication by a constant, addition and taking the modulus. It is also assumed that to each f of class T_0 there corresponds a number K, independent of p, such that

$$|f(p)| \leq K,$$

and that to each f there corresponds a finite "integral" $S(f)$ possessing the properties C, A, L, M, or an integral $I(f)$ having the properties C, A, L, P. If $U(f)$ is a functional operation on f these properties are

$$(C) \qquad U(cf) = cU(f),$$

$$(A) \qquad U(f_1 + f_2) = U(f_1) + U(f_2).$$

(L) If $f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n$, then

$$\lim U(f_n) = 0.$$

$$(P) \qquad U(f) \geq 0 \quad \text{if} \quad f \leq 0.$$

$$(M) \qquad |U(f)| \leq M(|f|).$$

It will be necessary to make frequent references to the earlier paper and these will be given by D 3(1), for example. We shall also use a notation which was used in that paper

$$f(p) = f_1(p) \vee f_2(p),$$

the logical sum of f_1, f_2, if, for each p, the value of f is the greater of the values of f_1, f_2. If the lesser value is chosen the combination is called the logical product

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* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), p. 279.

J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien (1913), p. 1295.

W. H. Young, Proceedings of the London Mathematical Society, vol. 13 (1914), p. 109.

$$f = f_1 \wedge f_2.$$

One particular theorem is so valuable that we restate it here for the convenience of the reader.

D. 7(7). If $f_1, f_2, \cdots$ is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leq \varphi$ for all n, f is summable, $\lim I(f_n)$ exists and $= I(f)$.

In the first paragraph we prove that an S-integral can be expressed in terms of a single I-integral, and use the concept of convergence in the mean. In the second paragraph measurable functions are defined, and it is proved that, under certain conditions, the fourth proportional of three measurable functions is measurable. The third paragraph deals with repeated integration and sequences of integrals. Finally we apply the previous analysis to special cases, more particularly to the generalized Stieltjes integral. Those who are more interested in the latter may turn immediately to the end of the paper.

1. **The S-integral as an I-integral.** In the previous paper D. 3. we defined $I_1(f)$ as the upper bound of $S(\varphi)$ for all functions φ of class T_0 such that $0 \leq \varphi \leq f$. $I_1(f)$ is called the positive integral, $I_2(f) = I_1(f) - S(f)$ the negative and $I(f) = I_1 + I_2$ the modular integral associated with S. Then $S(f) = I_1(f) - I_2(f)$, the difference of two I-integrals. Now in theorem 1(7) we prove that a function λ, equal to 1 or -1 everywhere, exists such that

$$S(f) = I(\lambda f).$$

1(1). Definition. A sequence of functions, f_n, is *convergent in the mean* with respect to I if the functions, f_n, are summable I and if

$$\lim_{m,\, n \doteq \infty} I(|fm - f_n|) = 0.$$

1(2). Definition. A sequence of functions, f_n, *converges in the mean to* f if f_n ,f are summable I, and if

$$\lim_{n \doteq \infty} I(|f - f_n|) = 0.$$

These two definitions can be proved to be equivalent; for if the conditions of 1(2) are satisfied, given any $e > 0$ we can find n_0 so that

$$I(|f_m - f_n|) \leq I(|f - f_m|) + I(|f - f_n|)$$
$$< e + e \qquad m, n \geq n_0.$$

The following theorem proves the converse.

1(3). If the sequence, f_n, is *convergent in the mean* there exists a summable function f to which the sequence converges in the mean. Given any $e > 0$ we can find n_0 so that

$$I(|f_m - f_n|) < e \quad m, n \geqq n_0.$$

Let $n_1, n_2, \cdots, n_s, \cdots$ be the numbers corresponding to e equal to 2^{-1}, $2^{-2}, \cdots, 2^{-s}, \cdots$ but picking them out so that $n_1 < n_2 < \cdots$. Then

$$I(|f_m - f_n|) < 2^{-s} \quad m, n \geqq n_s.$$

Let $f_{n_s} = g_s$ and

$$\varphi_{st} = g_s \vee g_{s+1} \vee \cdots \vee g_{s+t}.$$

At a particular p, φ_{st} will be one of the values of g_{s+r}, $r = 0, , \cdots t$. But

$$g_{s+r} = g_s + (g_{s+1} - g_s) + \cdots + (g_{s+r} - g_{s+r-1})$$
$$\leqq g_s + |g_{s+1} - g_s| + \cdots + |g_{s+r} - g_{s+r-1}|.$$

If we denote

$$\psi_{st} = g_s + |g_{s+1} - g_s| + \cdots + |g_{s+t} - g_{s+t-1}|,$$
$$\varphi_{st} \leqq \psi_{st},$$
$$I(\varphi_{st}) \leqq I(\psi_{st})$$
$$< I(g_s) + 2^{-s} + 2^{-s-1} + \cdots + 2^{-s-t+1}$$
$$< I(g_s) + 2^{-s+1}.$$

$\varphi_{s1} \leqq \varphi_{s2} \leqq \cdots$ is a nondecreasing sequence so that by D7(6) $\lim\limits_{t \doteq \infty} \varphi_{st} = \varphi_s$ exists, is summable I and

$$I(\varphi_s) \leqq I(g_s) + 2^{-s+1}.$$

Again

$$-\varphi_s \leqq \varphi_{st} \leqq -g_s$$
$$= -g_1 + (g_1 - g_2) + \cdots + (g_{s-1} - g_s)$$
$$\leqq -g_1 + |g_1 - g_2| + \cdots + |g_{s-1} - g_s|.$$
$$I(-\varphi_s) \leqq I(-g_1) + 2^{-1} + 2^{-2} + \cdots + 2^{-s+1}$$
$$\leqq I(-g_1) + 1.$$

But $-\varphi_1 \leqq -\varphi_2 \leqq \cdots$ is nondecreasing and by the same theorem $\lim\limits_{s \doteq \infty} (-\varphi_s) = -f$ exists, is summable and

$$I(f) = \lim_{s \doteq \infty} I(\varphi_s).$$

Given any positive e we can find s so that both $2^{-s+2} < e$ and

$$I(|f - \varphi_s|) < \tfrac{1}{4}e.$$

Then because $\varphi_s \geqq g_s$

$$I(|\varphi_s - g_s|) = I(\varphi_s) - I(g_s)$$
$$< 2^{-s+1} < \tfrac{1}{2}e.$$

If n_0 is the number n_s corresponding to the above s,

$$I(|g_s - f_n|) < 2^{-s} < \tfrac{1}{4}e \qquad n \geqq n_0.$$

Combining the three inequalities

$$I(|f - f_n|) < e \qquad n \geqq n_0.$$

This proves the theorem and therefore the definitions are equivalent.

Definition. Two functions are *nearly equal* with respect to I if the integral of their modular difference is zero. If f, g are two functions to which a sequence converges in the mean they must be "nearly equal," for given $e > 0$ we can find n_0, n_1 so that

$$I(|f - g|) \leqq I(|f - f_n|) + I(|g - f_n|$$
$$< 2e \qquad n \geqq n_0 \vee n_1.$$

1 (4) Definition. The *outer* measure of a set e of elements p is defined as the upper semi-integral $I(f)$ of the function equal to 1 on e and 0 otherwise. Since the above function is non-negative its lower semi-integral is non-negative; so that, if a set has a zero outer measure the corresponding function is summable and has the integral 0. In this case we say that the set is of zero measure.

Theorem. Two nearly equal functions are equal *nearly everywhere*, that is, except on a set of zero measure. For if their modular difference be called h, $h \geqq 0$ and $I(h) = 0$. If $E(e)$ is the set where $h \geqq e > 0$ and f_e the corresponding function, $ef_e \leqq h$ and $e\overline{I}(f_e) \leqq I(h) = 0$ by D. 6(3). Then $\underline{I}(f_e) \leqq \overline{I}(f_e) = 0 = I(f_e)$. Now as e approaches zero monotonously the sequence f_e is nondecreasing and of constant zero integral. If E is the set where $h > 0$, $f_E = \lim_{e \doteq 0} f_e$ and the measure of $E = I(f_E) = \lim I(f_e) = 0$.

1 (5). If $h \geqq 0$ is of class T_0 there exists a function k summable I such that $0 \leqq k \leqq h$ and

$$S(k) = I_1(h).$$

Since $I_1(h)$ is by definition the upper bound of $S(\varphi)$ $0 \leqq \varphi \leqq h$ given $e_i > 0$ we can find φ_i such that $0 \leqq \varphi_i \leqq h$ and

$$I_1(\varphi_i) - I_2(\varphi_i) = S(\varphi_i) > I_1(h) - e_i$$
$$I_1(h - \varphi_i) + I_2(\varphi_i) < e_i.$$

Similarly given $e_j > 0$ we can find φ_j so that

$$I_1(h - \varphi_j) + I_2(\varphi_j) < e_j.$$

But $|\varphi_i - \varphi_j|$ is not greater than $\varphi_i + \varphi_j$ and also not greater than $|h - \varphi_i| + |h - \varphi_j|$ so that

$$\begin{aligned} I(|\varphi_i - \varphi_j|) &= I_1(|\varphi_i - \varphi_j|) + I_2(|\varphi_i - \varphi_j|) \\ &< e_i + e_j. \end{aligned}$$

If $e_i \doteq 0, e_j \doteq 0$ as $i, j \doteq \infty$

$$\lim_{i,j \doteq \infty} I(|\varphi_i - \varphi_j)| = 0.$$

By definition 1(1), the sequence, φ_i, is convergent in the mean and by theorem 1(3) it converges in the mean to a summable function k. Recalling the method by which k is found and since $0 \leq \varphi_i \leq h$, $0 \leq k \leq h$ and

$$\begin{aligned} I_1(h - k) + I_2(k) &= \lim_{i \doteq \infty} I_1(h - \varphi_i) + I_2(\varphi_i) \\ &= 0. \end{aligned}$$

But both $I_1(h - k)$ and $I_2(k)$ are non-negative so that each is zero separately, and

$$S(k) = I_1(k) - I_2(k) = I_1(h) - 0 = I_1(h).$$

Since $0 \leq k \leq h$, k vanishes with h and we can find a function θ such that $0 \leq \theta \leq 1$ and $k = \theta h$. Then

$$S(\theta h) = I_1(h).$$

But $I_1(h - \theta h) = 0, I_2(\theta h) = 0$ so that $\theta = 1$ except on a set of I_1 – measure 0 and $\theta = 0$ except on a set of I_2 – measure 0; or $\theta = 0$ or 1 except on a set of I-measure 0.

1(6). If f is summable and vanishes with h

$$S(\theta f) = I_1(f).$$

It is sufficient to assume f non-negative for otherwise it is the difference of two non-negative functions.

If

$$f_n = f \wedge nh, \qquad I_1(f) = \lim I_1(f_n), \qquad S(\theta f) = \lim S(\theta f_n).$$

But

$$I_1(f_n) - I_1(\theta f_n) \leq nI_1(h - k) = 0$$

$$I_2(\theta f_n) \leq nI_2(k) = 0.$$

$$S(\theta f) = \lim S(\theta f_n) = \lim I_1(f_n) = I_1(f).$$

1(7). If $h \geq 0$ is summable S and limited there exists a function λ

which is everywhere equal to 1 or -1 and such that if f is summable and vanishes with h,

$$S(f) = I(\lambda f).$$

Extend the class T_0 to include h and combinations of h with members of T_0 so that the new class $T_0(+h)$ satisfies the required conditions for a class T_0. In obtaining $I_1(f)$ as the upper bound of $S(\varphi)$ we have to consider at least all the old functions φ with some additions. Then the new $I_1(f)$ $f \geq 0$ cannot be less than the old. On the other hand for all the new $\varphi \leq f$ as well as the old, $S(\varphi)$ is not greater than the old $I_1(f)$. The old and the new positive integrals and consequently the old and new modular integrals will be the same. Then all the way through the integrals will be identical. But in the new class $T_0(+h)$, h is a member and the previous theorems can be applied. Define $\theta' = \theta$ when $\theta = 1$ or 0 and $\theta' = 1$ otherwise, that is on a set of zero measure. Then $\theta' f$ is summable and

$$S(\theta' f) = I_1(f).$$

Let $\lambda = 2\theta' - 1 = 1$ or -1 everywhere. Then λf is summable and

$$\begin{aligned} f &= \lambda^2 f \doteq 2\theta'\lambda f - \lambda f, \\ S(f) &= S(2\theta'\lambda f) - S(\lambda f) \\ &= 2I_1(\lambda f) - S(\lambda f) \\ &= I(\lambda f). \end{aligned}$$

If it is possible to find a function $h > 0$ summable S this theorem would be true for all functions summable S. But in the general theory there may be no such function h vanishing nowhere and summable. We return to this question in the last paragraph.

2. **Measurable functions.** In the theory of Lebesgue integrals there is a valuable distinction between summable and measurable functions. In the general integral it is equally valuable and any function belonging to the Borel-extension of T_0 (extension by successive limiting processes and linear combinations) is measurable.

DEFINITION. If $h \geq 0$ is summable I and nowhere infinite, f is said to be measurable hI if the function $mh \vee f \wedge Mh$ is summable for all m, M $(m \leq M)$. This function is equal to mh when $f < mh$, to f when $mh \leq f \leq Mh$ and to Mh when $f > Mh$. Any summable function is measurable hI for every h.

$$mh \vee (f \vee g) \wedge Mh = (mh \vee f \wedge Mh) \vee (mh \vee g \wedge Mh)$$

so that by using $D7(5)$ the logical sum, and similarly the logical prdouct, of any two functions measurable hI is measurable hI. In particular if f is measurable so are $f \vee 0$, $-f \vee 0$. The converse is also true, that if $f \vee 0$, $-f \vee 0$ are measurable so is f. Define the function

$$\begin{aligned} \varphi(p, s, f) &= 0 \quad f(p) \leqq sh(p) \\ &= h(p) \quad f(p) > sh(p). \end{aligned}$$

2(1). The necessary and sufficient condition that f be measurable hI is that $\varphi(p, s, f)$ is summable for all s.

It is necessary, for if f is measurable hI, the function $f - sh$ is also measurable and $\varphi(p, s, f) = \varphi(p, 0, f - sh)$. We only require to prove that if f is measurable hI, $\varphi(p, 0, f)$ is summable. Also $f \vee 0$ is measurable and $\varphi(p, 0, f) = \varphi(p, 0, f \vee 0)$, and therefore we can assume f non-negative without loss of generality. Let $\varphi_n = nf \wedge h$ then φ_n is summable and since $nf \leqq (n+1)f$ (f being non-negative), $\varphi_1 \leqq \varphi_2 \leqq \cdots$ is a nondecreasing sequence of summable functions and $|\varphi_n| = \varphi_n \leqq h$ which is summable. If at $p, f = 0$, $\varphi_n = 0$ $(n = 1, 2, \cdots)$ or $\lim \varphi_n = 0$. On the other hand, if at $p, f > 0$, some n_0 exists depending possibly on p such that $nf > h$, $n \geqq n_0$ and then $\lim \varphi_n = h$. Therefore $\lim \varphi_n = \varphi(p, 0, f)$ is summable by D.7(7).

The condition is sufficient for considered as a function of s, $\varphi(p, s, f)$ is non-increasing and limited for each separate p. Consequently φ is integrable Riemann with respect to s in a limited interval. Consider the Riemann integral

$$\int_m^M \varphi(p, s, f) ds.$$

If $s_0 = m < s_1 < \cdots < s_n = M$ and if $s_i \leqq t_i < s_{i+1}$, the integral is defined as the limit of

$$\sum_i \varphi(p, t_i, f)(s_{i+1} - s_i),$$

as the maximum difference $s_{i+1} - s_i$ approaches zero. The above sum is a linear combination of summable functions and its modulus cannot exceed $(M - m)\varphi(p, m, f)$ which is summable so that by *D*.7(7) the limit function, that is, the function of p defined by the Riemann integral is summable. But this integral is exactly the function

$$mh \vee f \wedge Mh - mh.$$

Case 1. $h(p) = 0$, $\varphi(p, s, f) = 0$ for all s, the integral $= 0$ and so is the function designated.

Case 2. $f(p) > Mh(p)$, $\varphi(p, s, f) = h$ $(m \leqq s \leqq M)$ the integral $= Mh - mh$.

Case 3. $f(p) \leq mh(p)$, $\varphi(p, s, f) = 0$ $(m \leq s \leq M)$ and the integral $= 0 = mh - mh$.

Case 4. $f(p) = th(p)$, $m < t \leq M$ then $\varphi(p, s, f)$ equals h when $s < t$ and 0 beyond so that the integral becomes $(t - m)h = f - mh$. Then the function designated is equal to that defined by the integral and is summable; or $mh \vee f \wedge Mh$ is summable and f is measurable hI. If we make $m = 0$, we obtain at the same time the relation

$$f \vee 0 = \int_0^\infty \varphi(p, s, f)ds,$$

provided f vanishes with h. For then

$$\begin{aligned} 0 \vee f &= \lim_{M \doteq \infty} 0 \vee f \wedge Mh \\ &= \lim_{M \doteq \infty} \int_0^M \varphi(p, s, f)ds \\ &= \int_0^\infty \varphi(p, s, f)ds \end{aligned}$$

in the sense that when $0 \vee f$ is finite the equality is valid while if it is infinite, the integral is divergent to $+\infty$. The same theorem which proves that the above Riemann integral exists proves that if $f \geq 0$ is summable and vanishes with h,

$$I(f) = \int_0^\infty I[\varphi(p, s, f)]ds.$$

2(2). DEFINITION. If f is measurable hI and if a function $\theta(p)$ exists, finite for each p, such that $f = \theta h$, f is said to be *commensurable* hI. If f satisfies this condition and is also summable it is said to be *summable* hI.

THEOREM. If $f \geq 0$ is commensurable hI, either $\underset{\cdot}{I}(f) = \infty$ or f is summable hI.

For $f \wedge nh = f_n$ is summable hI and $f_1 \leq f_2 \leq \cdots \leq f = \lim f_n$. If then $\underset{\cdot}{I}(f)$ is finite, $I(f_n) \leq \underset{\cdot}{I}(f)$ so that by $D.7(6)$, $f = \lim f_n$ is summable.

In the following theorems of this paragraph there will enter some ratios of functions. We shall use the convention that *a meaningless fraction* of the form $\frac{0}{0}$ *is replaced by* 0.

2(3). If $f \geq 0$ is commensurable hI and λ is any constant index, $h(f/h)^\lambda$ is commensurable hI where the above convention is used and where in the case of several values the real non-negative value is chosen. Denote the function by g and $1/\lambda$ by μ. If $\lambda > 0$, when $g > sh$, $f > s^\mu h$

and when $g \leq sh$, $f \leq s^{\mu}h$ so that

$$\varphi(p, s, g) = \varphi(p, s^{\mu}, f)$$

and this is summable. Then by theorem 2(1) g is measurable hI.

$\varphi(p, s, f)$ is a monotone function of s, summable I and of modulus not greater than h. Hence

$$\varphi(p, s - 0, f) = \lim_{\epsilon \doteq 0,\ \epsilon > 0} \varphi(p, s - \epsilon, f)$$

is summable. But if $\lambda < 0$,

$$\varphi(p, s, g) = h(p) - \varphi(p, s^{\mu} - 0, f)$$

is summable. If $\lambda = 0$, $g = h$ is measurable hI. Again, if $f = \theta h$, $g = \theta^{\lambda}h$ so that g is commensurable hI.

2(4). If f, g are commensurable hI, so is $f + g$. The functions

$$\begin{aligned} \varphi(p, s - 0, g) &= 0 \quad g < sh \\ &= h \quad g \geq sh \end{aligned}$$

and $\varphi(p, s + 0, g) = \varphi(p, s, g)$ are summable. The product of two functions of type φ divided by h is also their logical product,

$$\varphi_1\varphi_2/h = \varphi_1 \wedge \varphi_2$$

and is summable. Also the function $\varphi(p, t, g)/h$ is a bounded non-increasing function of t and the generalized Stieltjes integral

$$-\int_{-\infty}^{+\infty} \varphi(p, s - t, f)d_t\varphi(p, t, g)/h$$

exists and the function so defined is summable I. The proof uses $D.7(7)$ as in theorem 2(1). At a particular p suppose that $g = t_1h$, then $\varphi(p, t, g)/h = 0$ if $t > t_1$ and 1 if $t \leq t_1$. Then the value of the integral (with the minus sign) will be

$$\varphi(p, s - t_1, f)[\varphi(p, t_1 - 0, g)/h - \varphi(p, t_1 + 0, g)/h] = \varphi(p, s - t_1, f).$$

That is to say, the function defined by the integral will be 0 or h according to whether $f \leq (s - t_1)h$ or $> (s - t_1)h$, or whether $f + g \leq sh$ or $f + g > sh$. Therefore the integral represents $\varphi(p, s, f + g)$ which must be summable from what was said before about the integral. Using theorem 2(1), $f + g$ is measurable hI. Also if $f = \theta_1h$, $g = \theta_2h$, $f + g = (\theta_1 + \theta_2)h$ and is commensurable hI.

Corollary. Any linear combination of functions commensurable hI is commensurable hI.

2(5). If f, g are commensurable hI so is fg/h, the fourth proportional of f, g and h. In succession using 2(4) Cor. and 2(1) we can show that $k = \frac{1}{2}(f + g)$, $l = \frac{1}{2}(f - g)$, $k \vee 0$, $-k \vee 0$, $l \vee 0$, $-l \vee 0$, and $h(k/h)^2 = h(k \vee 0)^2/h^2 + h(-k \vee 0)^2/h^2$, $h(l/h)^2$ are commensurable hI. Finally

$$fg/h = h(k/h)^2 - h(l/h)^2$$

is commensurable hI.

2(6). The limit of a convergent sequence of functions measurable hI is measurable hI. For if $f = \lim f_n$, at each element p

$$mh \vee f \wedge Mh = \lim (mh \vee f_n \wedge Mh).$$

$$|mh \vee f_n \wedge Mh| \leqq (|M| + |m|)h$$

which is summable; or, by D7(7), $mh \vee f \wedge Mh$ is summable and f is measurable hI.

Combining this theorem with 2(4) Cor., any function belonging to the Borel extension of the subclass of functions of T_0 commensurable hI, is also commensurable hI.

2(7). If f is measurable hI, it is measurable kI for any function $k \geqq 0$ which is nowhere infinite and which vanishes with h. The function

$$\begin{aligned} \varphi(p, s, f) &= 0 \quad f \leqq sh \\ &= h \quad f > sh, \end{aligned}$$

is summable and is a non-increasing bounded function of s. The same is true when k takes the place of f. The integral

$$-\int_{m-0}^{M+0} t\varphi(p, st, f)\,d_t\varphi(p, t, k)/h$$

exists and can be proved to be summable by a method similar to that used in the proof of theorem 2(1). If at a certain p, $k = ch$, $m \leqq c \leqq M$, the negative integral has the value

$$\begin{aligned} c\varphi(p, sc, f) &= 0 \qquad\quad f \leqq sch = sk \\ &= ch = k \quad f > sch = sk. \end{aligned}$$

On the other hand if c lies outside the range mM, $\varphi(p, t, k)/h$ is constant throughout the range and the integral is zero. The function defined by the integral is summable I for all m, M and in modular value does not exceed k which is summable. As a function of m or M it is monotone so that the limit as $m \doteq -\infty$, $M \doteq +\infty$ exists and is summable I. This limit is the function equal to 0 when $f \leqq sk$ and to k when $f > sk$, or it is the function $\varphi_1(p, s, f)$ using k as a base function in place of h. The theorem is therefore proved.

2(8) If f, g, k are commensurable hI is fg/k. For by 2(3), h^2/k is commensurable hI and by 2(5) fg/h is also. But

$$fg/k = (fg/h)(h^2/k)/h.$$

2(9) Define the function u equal to 1 when $h > 0$ and to 0 when $h = 0$. *If the function equal to 1 everywhere is measurable hI, u is commensurable hI.* Then h^λ is commensurable hI where λ is a constant index; for

$$h^\lambda = h(u/h)^{1-\lambda}.$$

Also if f, g are commensurable hI so also are f/g, fg; for

$$f/g = uf/g,$$

$$fg = fg/u.$$

3. Iterated integrals and sequences of integrals. 3(1). Let $S(q, f)$ be a collection of S-integrals on functions f of elements p, distinguished by the letters q which also refer to general elements not necessarily the same as p. Let these S-integrals be defined in terms of the same initial class T_0 of functions of p. Let Q be a positive I-integral on functions of q such that if f belongs to T_0, $S(q, f)$, $I(q, f)$ the modular integral associated with $S(q, f)$ are summable Q. For such functions f define

$$S(f) = Q[S(q, f)].$$

Then this equality holds for any function f belonging to the Borel extension of T_0 provided $I(q, |f|)$ is summable Q.

As at the beginning of paragraph 2 by the Borel extension is meant the extension by successive linear combinations and limiting processes. It is evident that the $S(f)$ so defined satisfies (C) and (A). Also

$$\begin{aligned} |S(f)| &\leq Q[|S(q, f)|] \\ &\leq Q[I(q, |f|)] \end{aligned}$$

so that the condition M is satisfied by S. Again if

$$f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n,$$

$I(q, f_n)$ is in modular value not greater than $I(q, f_1)$ which is summable Q. It follows from D7(7) that

$$\begin{aligned} |\lim S(f_n)| &\leq \lim Q[I(q, f_n)] \\ &= Q[\lim I(q, f_n)] \\ &= Q(0) = 0. \end{aligned}$$

Therefore $S(f)$ satisfies all the required conditions for an S-integral.

If $0 \leq \varphi \leq f$,

$$\begin{aligned} S(\varphi) &= Q[S(q, \varphi)] \\ &\leq Q[I_1(q, \varphi)] \\ &\leq Q[I_1(q, f)], \end{aligned}$$

and therefore the upper bound $I_1(f)$ satisfies

$$I_1(f) \leq Q[I_1(q, f)]$$

$$I_2(f) = I_1(f) - S(f) \leq Q[I_2(q, f)]$$

provided $f \geq 0$.

Now let f be a function of class T_1, the limit of a non-decreasing sequence, f_n, of functions of class T_0, such that $I(q, f)$ is summable Q, then since $|f_n| \leq |f| + |f_1|$,

$$\begin{aligned} I_1(f_n) &\leq Q[I_1(q, |f_n|)] \\ &\leq Q[I_1(q, |f|)] + Q[I_1(q, |f_1|)] \\ &\leq Q[I(q, |f|)] + Q[I(q, |f_1|)] \end{aligned}$$

which is finite. Then by $D7(6)$, $f = \lim f_n$ is summable I_1 and $I_1(f) = \lim I_1(f_n)$. Similarly $I_2(f) = \lim I_2(f_n)$, or f is summable S and

$$S(f) = \lim S(f_n) = \lim Q[S(q, f_n)].$$

But

$$|S(q, f_n)| \leq I(q, |f_n|) \leq I(q, |f|) + I(q, |f_1|)$$

which is summable Q, so that by D7(7)

$$\begin{aligned} S(f) &= Q[\lim S(q, f_n)] \\ &= Q[S(q, f)]. \end{aligned}$$

Evidently any linear combination of such functions f will again satisfy the equality and a similar proof may be used successively for one monotone limiting process after another, with linear combinations interpolated. Consider a more general limiting process $f = \lim f_n$ where the functions f_n already satisfy the equality, and where $I(q, |f|)$ is summable Q. We can choose f and f_n to be non-negative without loss of generality for $f \vee 0 = \lim (f_n \vee 0)$, $-f \vee 0 = \lim (-f_n \vee 0)$, $f = f \vee 0 - (-f \vee 0)$. Since $f = \lim f_n = \underline{\lim} f_n$, f is the limit of a non-decreasing sequence $g_1 \leq g_2 \leq \cdots \leq f$ where $g_n = f_n \wedge f_{n+1} \wedge \cdots$ is the limit of a non-increasing sequence, but all the functions involved lie between 0 and f inclusive so that proofs similar to that used above may be used to show that

$$S(g_n) = Q[S(q, g_n)],$$

$$S(f) = Q[S(q, f)],$$

in succession.

Before considering sequences of integrals we consider an operation which is a slight modification of the Moore general integral of which the limit of a sequence is a special case.

3(2). DEFINITION. $R(u)$, a functional operation on functions u of the elements q is said to be a *Moore positive integral* if there exists a class U of functions u which is equal to its star-extension,[*] which includes the modulus of each member of the class and such that for members of U the operation R satisfies

$$\text{(A)} \qquad R(u_1 + u_2) = R(u_1) + R(u_2),$$

$$\text{(C)} \qquad R(cu) = cR(u),$$

$$\text{(P)} \qquad R(u) \geq 0 \text{ if } u \geq 0.$$

If $u_1 \leq u_2$, $R(u_1) \leq R(u_2)$ since $R(u_2 - u_1) \geq 0$. If

$$\lim u_n = u\ ([q];\ U),$$

$$\lim R(u_n) = R(u).$$

For if the sequence, u_n, converges to u relative to the class U, that means that there exists a member $v(q)$ of U which we suppose replaced by its modulus such that for every $e > 0$ there is an n_e such that

$$|u - u_n| \leq ev \qquad n \geq n_e.$$

Then

$$\begin{aligned} |R(u) - R(u_n)| &= |R(u - u_n)| \\ &\leq R(|u - u_n|) \\ &\leq eR(v). \end{aligned}$$

3(3). DEFINITION. The collection $S(q, f)$ of S-integrals on functions f of the elements p and distinguished by the elements q is said to be *dominated by $J(f)$ with relative uniformity in $R(u)$* where R is a Moore positive integral if $S(q, f)$, J are defined in terms of the same class T_0 of functions of p, if for members of T_0, $S(q, f)$ belongs to the class U of functions of q, and if there exists a function $v(q)$ belonging to U, independent of f such that for each $f \geq 0$ of class T_0,

$$I(q, f) \leq v(q)J(f).$$

Here I is the modular integral associated with S, $J(f)$ is an I-integral. Corresponding to this definition we may say that a single S-integral $S(f)$ is dominated by $J(f)$ if they possess the same class T_0 and if a number M exists independent of f such that if $f \geq 0$ belongs to T_0,

$$I(f) \leq MJ(f),$$

when I is the modular integral associated with S.

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[*] E. H. Moore, Bulletin of the American Mathematical Society, vol. 18 (1912), p. 345.

3(4). **Theorem.** Under the conditions of 3(3)

$$S(f) = R[S(q, f)]$$

is dominated by J and the equality holds for all functions f summable J. Define the class $T_0(S)$ to be the class $T_0(J)$, then it is seen that for functions belonging to T_0, $S(f_1 + f_2) = S(f_1) + S(f_2)$, $S(cf) = cS(f)$ and $|S(f)| \leqq R(v)J(|f|)$. If $f \geqq 0$, $-f \leqq \varphi \leqq +f$.

$$S(\varphi) \leqq R(v)J(|\varphi|) \leqq R(v)J(f).$$

But $I(f)$ is the upper bound of such $S(\varphi)$ so that

$$I(f) \leqq R(v)J(f).$$

Therefore $S(f)$ is dominated by J. If $f_1 \geqq f_2 \geqq \cdots \geqq 0 = \lim f_n$,

$$\lim |S(f_n)| \leqq R(v) \lim J(f_n) = 0.$$

Therefore $S(f)$ satisfies all the required properties of an S-integral. S is dominated by J, any function summable J is summable S and the dominance inequality remains valid. For we can prove this first for f a member of T_1, summable J, and from this that if $f \geqq 0$ is summable J

$$\dot{I}(f) \leqq MJ(f).$$

But given $e > 0$ and a function f summable J we can find a member f of T_0 such that

$$J(|f - f_e|) < e/M.$$

$$\dot{I}(|f - f_e|) < e$$

so that f is summable I.

In this particular case $M = R(v)$,

$$\begin{aligned} |S(f) - S(f_e)| &\leqq I(|f - f_e|) \\ &\leqq R(v)J(|f - f_e|) \\ &< e, \\ |R[S(q,f)] - R[S(q,f_e)]| &\leqq R[I(q, |f - f_e)|] \\ &\leqq R(v)J(|f - f_e|) \\ &< e, \\ S(f_e) &= R[S(q, f_e)]. \end{aligned}$$

Therefore $S(f)$ differs from $R[S(q, f)]$ by less than $2e$ which may be as small as we please and consequently

$$S(f) = R[S(q, f)].$$

3(5). If a sequence, $S_n(f)$, of S-integrals is *uniformly dominated* by J and if, for every member of $T_0(J)$, $S_n(f)$ approaches a limit, this limit is an S-integral $S(f)$ and for every f summable J

$$S(f) = \lim S(f_n).$$

By "uniformly dominated" we mean that each S_n is dominated by J and that the corresponding numbers M_n are limited in their set.

This theorem can be regarded as a special case of 3(4). Take the numbers, n, to be the elements q, the class U to be the class of functions u_n for which $\lim u_n$ exists and let

$$R[u(n)] = \lim_{n \doteq \infty} u_n.$$

The function $v(q)$ will be taken to be a constant and equal to the upper bound of the numbers M_n. Then $R(v)$ = this upper bound.

4. The Stieltjes integral and other examples. We shall merely state some of the foregoing definitions and theorems as they are applicable to generalized Stieltjes (Radon-Young) integrals. Each is denoted by the letter a added to the number of the corresponding definition or theorem of which it is an application. Other applications can be supplied by the reader.

1(7a). If $\alpha(x, y)$ is of limited two-dimensional variation in $a \leqq x \leqq b$, $c \leqq y \leqq d$, and if $\omega(x, y)$ is its corresponding variation function, a functon $\lambda(x, y)$ = either 1 or -1 everywhere exists such that

$$\int_{a,c}^{b,d} f(x, y)\,d\alpha(x, y) = \int_{a,c}^{b,d} f(x, y)\lambda(x, y)\,d\omega(x, y)$$

for all functions $f(x, y)$ summable (ω). To apply theorem 1(7) choose $h(x, y) \equiv 1$. The one-dimensional case has been stated already by the author.*

3(1a). Let $\beta(x)$ be a limited non-decreasing function $a \leqq x \leqq b$; let $\alpha(x, s)$ be of limited variation in $s (c \leqq s \leqq d)$, the total variation being uniformly limited in x $(a \leqq x \leqq b)$, and for each s let it be limited and measurable Borel $(a \leqq x \leqq b)$. Then

$$\int^{d} f(s)\,d_s \int_a^b \alpha(x, s)\,d\beta(x) = \int_a^b \left[\int_c^d f(s)\,d_s\alpha(x, s) \right] d\beta(x),$$

provided $f(s)$ is limited and measurable Borel $c \leqq s \leqq d$. To apply 3(1) we choose $p = s$, $q = x$ and the functions of class T_0 to be the step-functions (constant over each of a finite set of subintervals into which (cd) is

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[*] P. J. Daniell, Transactions of the American Mathematical Society, vol. 19 (1918), p. 361.

divided). Such functions are linear combinations of functions of the type

$$f(s) = 1 \quad s_1 \leq s < s_2$$
$$= 0 \quad \text{otherwise}$$

and for this

$$\int_c^d f(s) d_s \int_a^b \alpha(x, s) d\beta(x) = \int_a^b \alpha(x, s_2 - 0) d\beta(x) - \int_a^b \alpha(x, s_1 - 0) d\beta(x)$$
$$= \int_a^b \left[\int_c^d f(s) d_s \alpha(x, s) \right] d\beta(x).$$

Again if $f(s)$ is limited and measurable Borel,

$$I(q, |f|) = \int_c^d |f(s)| d_s \omega(x, s)$$

where ω is the variation function corresponding to $\alpha(c \leq s \leq d)$

$$\leq T \max |f(s)|,$$

where T is the upper bound of the total variations of $\alpha(x, s)$, and since $\omega(x, s)$ is also measurable Borel $(a \leq x \leq b)$, $I(q, |f|)$ is summable β.

Corollary. This theorem can be immediately extended by allowing $\beta(x)$ to be a function of limited variation $(a \leq x \leq b)$. So extended the theorem is a generalization of one proved by Bray.*

3(5a). We say that a sequence, $\alpha_n(x)$ is uniformly Δ-bounded by a non-decreasing function $\gamma(x)$ if a number M exists independent of n, x_1, x_2 such that

$$|\alpha_n(x_2) - \alpha_n(x_1)| \leq M[\gamma(x_2) - \gamma(x_1)] \quad a \leq x_1 \leq x_2 \leq b.$$

Then the sequence of Stieltjes integrals,

$$\int_a^b f(x) d\alpha_n(x)$$

is uniformly dominated by $\int_a^b f(x) d\gamma(x)$. If the sequence $\alpha_n(x)$ is uniformly Δ-bounded by $\gamma(x)$ and if for each x it converges to $\alpha(x)$, $\alpha(x)$ is Δ-bounded by $\gamma(x)$ and

$$\int_a^b f(x) d\alpha(x) = \lim_{x \doteq \infty} \int_a^b f(x) d\alpha_n(x),$$

provided $f(x)$ is summable γ.

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* H. E. Bray, Annals of Mathematics, vol. 20 (1919), p. 183.

To apply 3(5) we choose J as the integral with respect to γ, T_0 the class of step-functions and since these are linear combinations of functions of the type

$$\begin{aligned} f(x) &= 1 \quad a \leqq x_1 \leqq x < x_2 \leqq b \\ &= 0 \quad \text{otherwise} \end{aligned}$$

it is sufficient to prove that

$$\lim\,[\alpha_n(x_2 - 0) - \alpha_n(x_1 - 0)] = \alpha(x_2 - 0) - \alpha(x_1 - 0).$$

Given $e > 0$ we can find $\delta > 0$ so that

$$\gamma(x_1 - 0) - \gamma(x_1 - \delta) < e/4M,$$

$$\gamma(x_2 - 0) - \gamma(x_2 - \delta) < e/4M.$$

Then $\alpha_n(x_2 - 0)$, $\alpha_n(x_1 - 0)$, $\alpha(x_2 - 0)$, $\alpha(x_1 - 0)$ all differ from $\alpha_n(x_2 - \delta)$, $\alpha_n(x_1 - \delta)$, $\alpha(x_2 - \delta)$, $\alpha(x_1 - \delta)$ by less than $\frac{1}{4}e$ which is independent of n. But

$$\lim\,[\alpha_n(x_2 - \delta) - \alpha_n(x_1 - \delta)] = \alpha(x_2 - \delta) - \alpha(x_1 - \delta)$$

so that $\lim\,[\alpha_n(x_2 - 0) - \alpha_n(x_1 - 0)]$ differs from $\alpha(x_2 - 0) - \alpha(x_1 - 0)$ by less than e.

This theorem may also be compared with a corresponding theorem of Bray's (loc. cit.) on the continuity of

$$\int_c^d f(s)d_s\alpha(x, s).$$

Other theorems such as 1(3), 2(3), 2(4), 2(5) can also be applied to generalized Stieltjes integrals. In all cases we choose $h \equiv 1$ as the basic summable function and T_0 as the class of step-functions.

The frequent appearance of the function $h \geqq 0$ summable I may seem unnecessary to some readers. In the usual applications of the general theory this h can be chosen equal to 1 and then it drops out of sight. But cases can be invented where this function would not be summable.

For example, let the elements p be the numbers $x (0 \leqq x \leqq 1)$, and let T_0 be the class of functions which are zero except at a finite number of values of x, that is of the type

$$\begin{aligned} f(x) &= f_i \quad x = x_i \quad i = 1, 2, \cdots n, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let $s(x)$ be some positive function finite for each x and define

$$I(f) = \sum_{i=1}^{n} f_i s(x_i).$$

This integral satisfies all the required conditions and at any particular x it is possible to find many members of T_0 which do not vanish at that point. Nevertheless there is no summable function $h \geqq 0$ which is greater than zero on a set having the power of the continuum. For then $sh > 0$ everywhere on this set, which is the limit of the set $sh > e$ as $e \doteq 0$ $(e > 0)$. The limit of a non-contracting sequence of sets which are of countable power is also of countable power so that a number $e > 0$ would exist such that $sh > e$ on a set of power greater than the countable. But then the integral would be greater than e added to itself a countable number of times which is infinite.

If difficulties arise even with so simple an example, it can be understood why in the general case there is frequent reference to a basic summable $h \geqq 0$.

Note. The author wishes to apologize for a statement in the paper of which this is a continuation. On page 279 it was suggested that Moore's use of relatively convergent sequences was a restriction and on page 281 it was stated that the Moore integral (which is again referred to here in 3(2)) was a special instance of the author's. The latter statement is untrue and the former misleading. In a sense there is a restriction because the class of functions to which Moore's integral can be applied is given initially and is not progressively extended to so wide a class, but on the other hand the restrictions placed on the integral operation of Moore are not so stringent. As in 3(5) the sum of a convergent series is a form of Moore integral. Our theory would consider only absolutely convergent series in the corresponding case.

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